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# Measures on the Splitting Subspaces of an Inner Product Space

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Let $S$ be an inner product space and let $E(S)$ (resp. $F(S)$) be the orthocomplemented poset of all splitting (resp. orthogonally closed) subspaces of $S$. In this article we study the possible states/charges that $E(S)$ can admit. We first prove that when $S$ is an incomplete inner product space such that $\dim \overline{S}/S < \infty$, then $E(S)$ admits at least one state with a finite range. This is very much in contrast to states on $F(S)$. We then go on showing that two-valued states can exist on $E(S)$ not only in the case when $E(S)$ consists of the complete/cocomplete subspaces of $S$. Finally we show that the well known result which states that every regular state on $L(H)$ is necessarily $\sigma$-additive cannot be directly generalized for charges and we conclude by giving a sufficient condition for a regular charge on $L(H)$ to be $\sigma$-additive.

**KEY WORDS:** Hilbert space; inner product space; splitting subspace; orthogonally closed subspace; state; charge.

## 1. INTRODUCTION

Let $S$ be an inner product space (real, complex, or quaternion). Unless otherwise stated, we shall not assume that $S$ is complete. For any subspace $M \subset S$ denote by $M^\perp$ the subspace of $S$ consisting of all the vectors that are orthogonal to $M$, i.e. $M^\perp = \{x \in S : \langle x, y\rangle = 0 \text{ for all } y \in M\}$. If $M$ and $N$ are any two subspaces of $S$ such that $M \subset N$, then we set $M^{\perp_N} = M^\perp \cap N$. For any subspace $M \subset S$ denote by $\overline{M}$ the completion of $M$, and if $M \subset N \subset S$, then, let us agree to denote by $\overline{M}^N$ the closure of $M$ in $N$, i.e. $\overline{M}^N = \overline{M} \cap N$. When it is known (or assumed) that $S$ is complete, i.e. that $S$ is a Hilbert space, we are usually writing $H$ instead of $S$. In addition, for any nonzero vector $x \subset S$, let $[x]$ denote the one-dimensional subspace of $S$ spanned by $x$.

We can define a number of families of closed subspaces of $S$. The most important examples with respect to the mathematical foundation of Quantum Mechanics are:

$$E(S) = \{M \subset S : M \oplus M^\perp = S\}, \quad \text{and} \quad F(S) = \{M \subset S : M^{\perp\perp} = M\},$$

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the system of splitting subspaces and the system of orthogonally closed subspaces of $S$, respectively.

Observe that $E(S) \subset F(S)$, and when $S$ is complete we have $E(S) = F(S)$ $(= L(S))$. Surprisingly enough, Amemiya and Araki (1996) proved that the converse is also true; i.e. if every orthogonally closed subspace of $S$ is splitting, then $S$ is complete. Indeed, it was shown that if $F(S)$ is orthomodular,[2] then $E(S) = F(S)$. The importance of this result stems from the fact that in general it is very unusual that an algebraic condition implies topological completeness.

Of great physical importance are measures defined on $E(S)$ and $F(S)$. A *charge* $m$ on $E(S)$ is a mapping $m : E(S) \to \mathbb{R}$ such that $m(A \vee B) = m(A) + m(B)$ whenever $A \subset B^{\perp}$. Charges on $F(S)$ are defined in a similar way. A state is a normalized positive charge. A charge $m$ on $E(S)$ (or $F(S)$) is said to be $\sigma$-additive if for every countable collection $\{M_i : i \in \mathbb{N}\}$ of mutually orthogonal elements in $E(S)$ (resp. $F(S)$), satisfying that $\vee_{i\in\mathbb{N}} M_{\mathrm{i}}$ exists in $E(S)$,[3] we have

$$m\left(\bigvee_{i\in\mathbb{N}} M_i\right) = \sum_{i\in\mathbb{N}} m(M_i). \tag{1.1}$$

(A charge is completely additive if Eq. (1.1) holds for every collection of mutually orthogonal subspaces.)

In Dvurečenskij and Pták (2002), the possible range that a state on $F(S)$ can have was investigated. It was shown that the range of a state on $F(S)$ is always the unit interval [0, 1]. This result was later extended in Chetcuti and Dvurečenskij (2003) for bounded charges and it was also shown that the range of unbounded, sign-preserving charges[4] satisfying the Jauch-Piron property is always the whole real line $\mathbb{R}$.

## 2. RANGE OF STATES ON *E*(*S*)

Every state $s$ on $F(S)$ must satisfy Range$(s) = [0, 1]$. The same cannot be said for states on $E(S)$. As the following theorem states, the different algebraic structure of $E(S)$ (see, for example Dvurečenskij, 1992) allows $E(S)$ to admit states taking only finitely many different values.

**Theorem 2.1.** *Let $S$ be an inner product space such that $0 < \dim \overline{S}/S = n < \infty$. Then $E(S)$ admits a state taking at most $n + 1$ values and vanishing on each complete subspace of $S$.*

[2] $F(S)$ is said to be orthomodular if for every $M, N \in F(S)$, $M \subset N$, we have $N = M \vee (M^{\perp} \wedge N)$.

[3] Observe that $F(S)$ is a complete lattice and therefore $\vee_{i\in\mathbb{N}} M_i$ always exists in $F(S)$.

[4] A charge $m$ on $F(S)$ is said to satisfy the *sign-preserving property* if for any countable collection $\{N_i : i \in \mathbb{N}\}$ of orthogonal finite-dimensional subspaces in $F(S)$ satisfying $m(N_i) > 0$, (resp. $m(N_i) < 0$) for all $i \in \mathbb{N}$, it follows that $m(\vee_{i\in\mathbb{N}} N_i) \geq 0$, (resp. $m(\vee_{i\in\mathbb{N}} N_i) \leq 0$).

**Proof:** Consider the mapping

$$s : E(S) \to \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\} \tag{2.1}$$

$$M \mapsto \frac{\dim \bar{M}/M}{n}. \tag{2.2}$$

(Observe that $\dim \bar{M}/M \leq n$ for all $M \in E(S)$). Let $M, N \in E(S)$ such that $M \perp N$. We show that $\dim \bar{M}/M + \dim \bar{N}/N = \dim \overline{(M \oplus N)}/(M \oplus N)$. Let $m_1 = \dim \bar{M}/M$ and $m_2 = \dim \bar{N}/N$ and let $\{x_1, x_2, \ldots, x_{m_1}\} \subset \bar{M}/M$, and $\{y_1, y_2, \ldots y_{m_2}\} \subset \bar{N}/N$ such that the systems $\{x_i + M : 1 \leq i \leq m_1\}$ and $\{y_i + N : 1 \leq i \leq m_2\}$ form bases in $\bar{M}/M$ and $\bar{N}/N$ respectively. We show that $\mathfrak{R} = \{x_i + (M \oplus N) : 1 \leq i \leq m_1\} \cup \{y_i + (M \oplus N) : 1 \leq i \leq m_2\}$ forms a basis in $\overline{M \oplus N)}/(M \oplus N) = (\bar{M} \oplus \bar{N})/(M \oplus N)$. If $w \in \bar{M} \oplus \bar{N}$, then $w = x + y$ for some $x \in \bar{M}$ and $y \in \bar{N}$. This implies that for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_{m1}, \beta_1, \beta_2, \ldots, \beta_{m2}$ and $u \in M, v \in N$, we have

$$x = \sum_{i \leq m_1} \alpha_i x_i + u,$$

$$y = \sum_{i \leq m_2} \beta_i y_i + v,$$

and hence, $x + y = \sum_{i \leq m_1} \alpha_i x_i + \sum_{i \leq m_2} \beta_i y_i + u + v$. This means that $\mathfrak{R}$ is spanning in $(\bar{M} \oplus \bar{N})/(M \oplus N)$. In addition, it is not difficult to show that $\mathfrak{R}$ is a linearly independent subset of $(\bar{M} \oplus \bar{N})/(M \oplus N)$. Hence $\dim \overline{(M \oplus N)}/(M \oplus N) = m_1 + m_2$. □

In the following example, we exhibit an incomplete inner product space $S$ such that $\dim \bar{S}/S = n < \infty$ and $E(S)$ admits a state $s$ satisfying

$$\text{Range}(s) = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}.$$

(As it will be shown in Remark 2.9, we can have $\dim \bar{S}/S = n \, (n \geq 2)$ and for each $M \in E(S)$, $\dim \bar{M}/M \in \{0, n\}$.)

*Example 2.2.* For $1 \leq i \leq n$, let $S_i$ be an incomplete dense hyperplane of a separable Hilbert space $H_i$ and let $S$ be the direct sum of $S_1, S_2, \ldots, S_n$, i.e.

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_n.$$

It is clear that $\dim \bar{S}/S = n$, and if we let $s : E(S) \to \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, defined by $s(M) = \frac{\dim \bar{M}/M}{n}$, then $s$ is *a* state on $E(S)$. What remains to show is that s is onto. Let $1 \leq k \leq n$ and let $M = S_1 \oplus S_2 \oplus \cdots \oplus S_k$. Of course, $M \in E(S)$, and moreover, $\dim \overline{M}/M = k$. Hence $s(M) = \frac{k}{n}$.

We now show that if $\dim \overline{S}/S = 2$, and if there exists an $M \in E(S)$ such that $\dim \overline{M}/M = 1$, then we can define plenty of states vanishing on each complete subspace of $S$. We shall need to introduce the following notion and then prove Proposition 2.3.

We define a mapping $\alpha : E(S) \to \mathcal{V}(\overline{S}/S)$, where $\mathcal{V}(\overline{S}/S)$ is the system of all subspaces of $\overline{S}/S$, by

$$\alpha(M) := \{\hat{x} : x \in \overline{M}\}, \quad M \in E(S),$$

where $\hat{x}$ denotes the class in $\overline{S}/S$ determined by a vector $x$. Observe that $\dim \alpha(M) = \dim \overline{M}/M$.

**Proposition 2.3.** *The mapping α satisfies the following properties.*

*(i) If $M, N \in E(S), M \subset N$, then $\{0\} = \alpha(\{0\}) \subset \alpha(M) \subset \alpha(N) \subset \alpha(S) = \overline{S}/S$.*
*(ii) If $M, N \epsilon E(S), M \perp N$, then $\alpha(M) \cap \alpha(N) = \{\hat{0}\}$.*
*(iii) If $M, N \subset E(S), M \perp N$, then $\alpha(M) + \alpha(N) = \alpha(M + N)$.*
*(iv) $\alpha(M) + \alpha(M^\perp) = \overline{S}/S$.*

**Proof:** $\alpha(M)$ is a linear subspace of $\overline{S}/S$.

(i) It is evident.
(ii) Let $\hat{x} \in \alpha(M) \cap \alpha(N)$. There are $x_1 \in \overline{M}$ and $x_2 \in \overline{N}$ such that $\hat{x} = \hat{x_1} = \hat{x_2}$. Hence $x_1 - x_2 = y$ for some $y \in S$. Then $y = y_1 + y_2 + y_3$, where $y_1 \in M$ and $y_2 \in N$, and $y_3 \in (M + N)^\perp$. Consequently, $x_1 - y_1 = x_2 + y_2 + y_3 \in \overline{M^\perp}$ and $x_1 - y_1 \in \overline{M}$ which yields $x_1 = y_1 \in M$, i.e. $\hat{x} = \hat{0}$.
(iii) It is clear that $\alpha(M) + \alpha(N) \subset \alpha(M + N)$. Let $\hat{x} \in \alpha(\overline{M} + N)$. Then $x \in \overline{M + N} = \overline{M + N}$ and $x = x_1 + x_2$, where $x_1 \in \overline{M}$ and $x_2 \in N$. Hence, $\hat{x} = \hat{x_1} + \hat{x_2}$, so that $\hat{x} \in \alpha(M) + \alpha(N)$.
(iv) It follows from (iii). □

*Example 2.4.* Suppose that $\dim \overline{S}/S = 2$ and that there exists a subspace $M \in E(S)$ with $\dim \overline{M}/M = 1$. We denote by $\mathcal{E}$ the system of couples $(\alpha(M), \alpha(M^\perp))$ such that $M \in E(S)$, $\dim \overline{M}/M = 1$, and if $(\alpha(M), \alpha(M^\perp)) \in \mathcal{E}$ then $(\alpha(M^\perp), (\alpha(M)) \notin \mathcal{E}$. Let $\{(\alpha M), \alpha(M^\perp))_\theta\}_{\theta\in\Theta}$ be any labelling of $\mathcal{E}$. Let $\pi_1$ and $\pi_2$ be the projections from $(\alpha(M), \alpha(M^\perp))$ to the first and second coordinate, respectively. We choose a family $\{p_\theta : \theta \in \Theta\}$ of real numbers from the unit interval $[0, 1]$.

We define a mapping $s : E(S) \to [0, 1]$ by

$$s(M) = \begin{cases} 0 & \text{if } \dim \bar{M}/M = 0, \\ 1 & \text{if } \dim \bar{M}/M = 2, \\ p_\theta & \text{if } \dim \bar{M}/M = 1, \pi_1((\alpha(M), \alpha(M^\perp))_\theta = \alpha(M), \\ 1 - p_\theta & \text{if } \dim \bar{M}/M = 1, \pi_2((\alpha(M^\perp), \alpha(M))_\theta = \alpha(M), \end{cases}$$

where $M \in E(S)$. Then $s$ is a state on $E(S)$ which vanishes on each complete subspace of $S$. In particular, if $p_\theta = 1/2$ for any $\theta \in \Theta$, we have the state given by Eq. (2.1).

In the following, we show that when $S$ is an inner product space with a countable linear dimension, then every state $s$ on $E(S)$ satisfies Range$(s) = [0, 1]$. The following lemma is a direct consequence of Gleason's Theorem (Dvurečenskij, 1992; Gleason, 1957). For the proof of the lemma, the reader is referred to (Dvurečenskij and Pták, 2002) Proposition 2.4.

**Lemma 2.5.** *Let $H_n$ be an n-dimensional Hilbert space, $n \geq 3$, and let s be a state on $L(H_n)$. Then either we have $s([x]) = \frac{1}{n}$ for all $x \in H_n$ $(x \neq 0)$, or*

$$[\min_{x \neq o} s([x]), \max_{x \neq 0} s([x])] \subset \text{Range}\,(s).$$

**Theorem 2.6.** *Let S be an inner product space with linear dimension equal to $\aleph_0$. Every state s on E(S) satisfies* Range$(s) = [0, 1]$.

**Proof:** Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal linear basis of $S$ and let $M = \text{span}\{e_{2i} : i = 1, 2, 3, \ldots\}$. Then $M^\perp = \text{span}\{e_{2i-1} : i = 1, 2, 3, \ldots\}$. We either have $s(M) \geq \frac{1}{2}$ or $s(M^\perp) \geq \frac{1}{2}$; it is harmless to assume the first. For any $n \geq 3$, we can express the set of all odd positive integers in the form of a disjoint countable union of $(n-1)$-element sets $I_j$, $j \in \mathbb{N}$. Put $H_j = \text{span}\{e_k : k \in I_j\} \oplus [e_{2j}]$ and let $K_n$ be an $n$-dimensional Hilbert space. Fix any $u \in \mathcal{S}(K_n)$, and for each $j \in \mathbb{N}$, let $U_j : K_n \to H_j$ be a unitary operator such that $U_j(u) = e_{2j}$. Define the map $\phi : L(K_n) \to E(S)$, $M \mapsto \text{span}\{\bigcup_{j \in \mathbb{N}} U_j M\}$. It is not difficult to verify that $\phi$ is well-defined (i.e. $\phi(M) \in E(S)$ for every $M \in L(K_n)$) and that if $M \perp N$ in $L(K_n)$, then $\phi(M) \perp \phi(N)$, and $\phi(M \oplus N) = \phi(M) \oplus \phi(N)$. Moreover, $\phi([u]) = M$. We can now define a state $\tilde{s}$ on $L(K_n)$ by $\tilde{s}(M) = s(\phi(M))$. Observe that $\tilde{s}([u]) = s(M) \geq \frac{1}{2}$. Certainly, there exists $v \in \mathcal{S}(K_n)$ such that $\tilde{s}([v]) \leq \frac{1}{n}$. Lemma 2.5 implies that $[\frac{1}{n}, \frac{1}{2}] \subset \text{Range}(s)$. We can repeat this for every $n \geq 3$ and thus obtain that $[0, \frac{1}{2}] \subset \text{Range}(s)$. By considering complements, we get $[0, 1] \subset \text{Range}(s)$. □

In Chetcuti (2002), there is an attempt to characterize inner product spaces $S$ for which $E(S)$ admits a two-valued state. It was not known whether the existence of a two-valued state on $E(S)$ implies automatically that $S$ is an incomplete hyperplane of $\overline{S}$. Here we give a negative answer to this question. Indeed, we show that for any $\eta \in \{1, 2, 3 \ldots\} \cup \{\aleph_0, 2^{\aleph_0}\}$, there exists an inner product space $S$ such that $\dim \overline{S}/S = \eta$ and $E(S)$ admits a two-valued state. First we prove the following lemma which follows the same lines of Lemma 2.2.3 in Pták and Weber (2001).

**Lemma 2.7.** *Let $S_1 \subset S_2$ be two inner product spaces such that $S_1$ is dense in $S_2$ and $\dim S_2/S_1 = 1$. Then for every $M \in E(S_1)$, at least, either $M$ or $M^{\perp_{S_1}}$ is closed in $S_2$.*

**Proof:** Suppose that $M \in E(S_1)$ such that neither $M$ nor $M^{\perp_{S_1}}$ is closed in $S_2$. Let $x \in \overline{M}^{S_2} \backslash M$ and $y \in \overline{M^{\perp_{S_1}}}^{S_2} \backslash M^{\perp_{S_1}}$. Since $\dim S_2/S_1 = 1$, there exist scalars $\alpha, \beta$ such that $\alpha x + \beta y = s$ for some $s \in S_1$. But $s = s_1 + s_2$, where $s_1 \in M$ and $s_2 \in M^{\perp_{S_1}}$. Then we have that $s_1 - \alpha x = \beta y - s_2$, which is a contradiction. □

**Theorem 2.8.** *For every $\eta \in \{1, 2, 3, \ldots\} \cup \{\aleph_0, 2^{\aleph_0}\}$, there exists an inner product space $S$ such that $\dim \overline{S}/S = \eta$ and E(S) admits a two-valued state.*

**Proof:** Let $H$ be an infinite-dimensional, separable Hilbert space and define $\zeta$ as follows: $\zeta = \eta - 1$ if $\eta \in \{1, 2, 3, \ldots\}$, and $\zeta = \eta$ if $\eta \in \{\aleph_0, 2^{\aleph_0}\}$. Let $S'$ be a dense subspace of $H$ having linear dimension equal to $2^{\aleph_0}$ such that $\dim H/S' = \zeta$. Let $\mathfrak{U}$ denote the collection of all the closed subspaces of $S'$ having a linear dimension equal to $2^{\aleph_0}$. It is not difficult to verify that $|\mathfrak{U}| = 2^{\aleph_0}$. Hence, we can express as $\mathfrak{U} = \{U_\alpha : 0 \le \alpha < \omega\}$, where $\omega$ is the first ordinal number with cardinality $2^{\aleph_0}$. Using transfinite induction, we can construct a linearly independent set of unit vectors $V = \{v_\alpha : 0 \le \alpha < \omega\} \subset S'$, such that $v_\alpha \in U_\alpha$ for each $\alpha$. We can extend this set to a linear basis (consisting of unit vectors) $K$ of $S'$. Expressing the set $\{p \in \mathbb{R} : p > 0\}$ as $\{p_\alpha : 0 \le \alpha < \omega\}$, we can define a linear functional $f$ on $S'$ by setting $f(v_\alpha) = p_\alpha$ for each $v_\alpha \in V$ and $f(v) = 0$ for all $v^- \in K \backslash V$. Let $S = \mathrm{Ker}(f)$. Then $S$ is dense in $S'$ and $\dim S'/S = 1$.

By the construction of $S$, and by Lemma 2.7, it follows that for all $M \in E(S)$, either $M$ or $M^{\perp_S}$ has a linear dimension less than $2^{\aleph_0}$. The mapping $s : E(S) \to \{0, 1\}$ defined by $s(M) = 0$ if the linear dimension of $M$ is less than $2^{\aleph_0}$, and $s(M) = 1$ if linear dimension of $M$ is $2^{\aleph_0}$ defines a two-valued state on $E(S)$. Observe that $\dim \overline{S}/S = \dim H/S = \eta$. The proof is complete. □

*Remark 2.9* In the case when $\eta < 2^{\aleph_0}$, we remark that from the construction of $S$, it follows that $E(S)$ merely consists of the finite/cofinite dimensional subspaces. This means that the state defined on $E(S)$ by (2.1) gives only a two-valued state. Indeed, in such case, this is the only state on $E(S)$ having a discrete range.

Let $C(S)$ be the collection of all complete and all cocomplete subspaces of $S$. (A subspace $B \subset S$ is said to be cocomplete if there exists a complete subspace $A \subset S$ such that $B = A^{\perp}$.) It is very easy to check that we always have the inclusion $C(S) \subset E(S)$. By Lemma 2.7 it follows that if $S$ is an incomplete hyperplane of $\overline{S}$, then $C(S) = E(S)$. Moreover, when $C(S) = E(S)$, then $E(S)$ admits a two-valued state. As the following example illustrates, the converse of this last statement is not true.

*Example 2.10* Let $H_1$ and $H_2$ be two separable Hilbert spaces, and let $\{e_i : i \in \mathbb{N}\}$ be an ONB of $H_1$. Put $S_0 = \text{span}\{e_i : i \in \mathbb{N}\}$ and define $S'$ to be the direct sum of $S_0$ and $H_2$, i.e. $S' = S_0 \oplus H_2$. Now we apply the technique used in the proof of Theorem 2.8 to derive a dense hyperplane $S$ of $S'$ such that $S_0 \subset S$ and $E(S)$ admits a two-valued state. If we let $\omega$ to be the first ordinal number with cardinality $2^{\aleph_0}$, and $\{U_\alpha \subset S' : 0 \leq \alpha < \omega\}$ to be the collection of closed subspaces of $S'$ having a linear dimension equal to $2^{\aleph_0}$, then we can use transfinite induction and construct a linearly independent set of unit vectors $V = \{v_\alpha \in S' : 0 \leq \alpha < \omega\}$ such that:

(i) $v_\alpha \in U_\alpha$ for each $\alpha$,
(ii) $\{e_i : i \in \mathbb{N}\} \cup V$ is a linearly independent set in $S'$.

Now we proceed exactly as in the proof of Theorem 2.8, and we extend $\{e_i : i \in \mathbb{N}\} \cup V$ to a linear basis $K$ of $S'$. After expressing the set of positive reals as $\{p_\alpha : 0 \leq \alpha < \omega\}$, we can define an unbounded linear functional $f$ on $S'$ in such a way that it is vanishing on all the vectors in $K \backslash V$, and such that $f(v_\alpha) = p_\alpha$. It is clear that if we let $S = \text{Ker}(f)$, then $E(S)$ admits a two-valued state. Moreover, observe that $S_0 \in E(S)$, and $S_0$ is neither complete nor cocomplete.

## 3. REGULAR STATES ON *E*(*S*)

In this section we study regular states on the system of splitting subspaces of an infinite-dimensional inner product space $S$. A charge $m$ on $E(S)$ is said to be *regular* if for every $\epsilon > 0$ and $A \in E(S)$, there exists a finite-dimensional subspace $M \subset A$ such that $|m(A) - m(M)| < \epsilon$. For any charge $m$ on $E(S)$, we set

$$\text{Range}_f(m) := \{m(A) : A \in E(S), \dim A < \infty\},$$

and for any integer $n = 0, 1, \ldots,$ we set

$$\text{Range}_n(m) := \{m(A) : A \in E(S), \dim A = n\}.$$

First we consider the case when $S$ is complete, i.e. we consider regular states on $L(H)$, where $H$ denotes an infinite-dimensional Hilbert space. The range of every state on $L(H)$ is [0, 1] (Dvurečenskij and Pták, 2002). Every regular state $s$ on $L(H)$ is of the form $s(M) = \text{tr}(TP_M)$, where $T$ is a Hermitian trace operator on

$H$ with unit trace (Dvurečenskij, 1992). This implies that $s$ is completely additive. When $H$ is separable, every $\sigma$-additive state on $L(H)$ is regular (see also Theorem 4.1), and therefore, the regular states on $L(H)$ are precisely the ones that are $\sigma$-additive.

**Theorem 3.1.** *Let H be an infinite-dimensional Hilbert space and let T be a positive Hermitian trace operator on H with unit trace. The state $s_T$ on L(H) defined by $s_T(M) = \mathrm{tr}(TP_M)$, $M \in L(H)$, satisfies $[0, 1) \subset \mathrm{Range}_f(s_T)$. Moreover, $1 \in \mathrm{Range}_f(s_T)$ if, and only if, T has a finite system of proper vectors.*

**Proof:** Hermitian trace operators can be expressed in the form

$$T = \sum_{\imath\in i} \lambda_i x_i \otimes \overline{x_i}, \tag{3.1}$$

where $\{\lambda_i : i \in I\}$ are the eigenvalues of $T$ (possibly repeated) corresponding to the proper vectors $\{x_i : i \in I\}$. Moreover, since $T$ is of unit trace, we have $\sum_{i\in I} \lambda_i = 1$. If $I$ is finite, we can find a finite ONS $\{u_i : i \in I\}$ such that $x_i \perp u_j$ for all $i, j \in I$. For any $\phi \in [0, \frac{\pi}{2}]$ and $i \in I$, we can then define $y_i = \cos\phi x_i + \sin\phi u_i$. Let $Y = \oplus_{i\in I}[y_i]$ and consider $s_T(Y)$,

$$\begin{aligned} s_T(Y) &= \sum_{i\in I} s_T([y_i]) = \sum_{i,j\in I} \lambda_j |\langle y_i, x_j\rangle|^2 \\ &= \sum_{i\in I} \lambda_i |\langle y_i, x_i\rangle|^2 = \sum_{i\in I} \lambda_i \cos^2\phi = \cos^2\phi. \end{aligned}$$

Hence, for $n = |I|$, $\mathrm{Range}_n(s_T) = \mathrm{Range}_f(s_T) = [0, 1]$.

Now suppose that $I$ is infinite. Given any $\epsilon > 0$, there exists a finite subset $I_0 \subset I$ such that $\sum_{i\in I_0} \lambda_i > 1 - \epsilon$. Let $\{u_i : i \in I_0\}$ be an ONS in $H$ such that $x_i \perp u_j$ for all $i, j \in I_0$.

Define, as in the first part of the present proof, $y_i = \cos\phi x_i + \sin\phi u_i$, $i \in I_0$, and let $Y_{I_0} = \oplus_{i\in I_0}[y_i]$. Then

$$\begin{aligned} s_T(Y_{I_0}) &= \sum_{i\in I_0}([y_i]) = \sum_{i\in I_0}\sum_{j\in I} \lambda_j |\langle y_i, x_j\rangle|^2 \\ &\geq \sum_{i,j\in I_0} \lambda_j |\langle y_i, x_j\rangle|^2 = \sum_{i\in I_0} \lambda_i \cos^2\phi = \cos^2\phi \sum_{i\in I_0} \lambda_i. \end{aligned}$$

This implies that $\mathrm{Range}_f(s_T) \supset [0, 1)$. Observe that $s_T(M) = 1$ if, and only if, $\{u_i : i \in I\} \subset M$, which yields that $1 \notin \mathrm{Range}_f(s_T)$ when $I$ is infinite. □

**Corollary 3.2.** *Let H be an infinite-dimensional Hilbert space and let s be a regular state on L(H). For any $A \in L(H)$, $\dim A = \infty$, we have*

$$[0, s(A)) \subset \{s(M) : M \subset A, \dim A < \infty\} \subset [0, s(A)].$$

We now recall to the fact that every state $s$ on $E(S)$ can be uniquely expressed in the form

$$s = \alpha s_R + \beta s_V,$$

where $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$, $s_R$ is a regular state and $s_V$ is a state on $E(S)$ vanishing on all the finite-dimensional subspaces of $S$. This was originally proved by Aarnes (1970) for the case when $S$ is a Hilbert space, and then generalized for all inner product spaces by the second author of this article (Dvurečenskij, 1991).

**Corollary 3.3.** *Let* $s = \alpha s_R + \beta s_V$ *be a state on* $L(H)$, $\dim H = \infty$. *Then* Range (s) = [0,1], *and*

$$[0, \alpha) \subset \text{Range}_f(s) \subset [0, \alpha]. \tag{3.2}$$

**Proof:** By the result proved in Dvurečenskij and Pták (2002), we have Range $(s) = [0, 1]$, and Corollary 3.2 implies Eq. (3.2). □

Now we consider states on $E(S)$ when $S$ is an incomplete inner product space. If $T$ is a positive Hermitian trace operator on $\overline{S}$ with unit trace, then the mapping $s_T$ on $E(S)$ defined by

$$s_T(M) = \text{tr}(T P_{\bar{M}}), \qquad M \in E(S), \tag{3.3}$$

is a regular state on $E(S)$. The converse is also true. (The reader may need to refer to Dvurečenskij (1992), Theorem 4.3.5.) Indeed, every regular state on $E(S)$ is of the form defined by Eq. (3.3) for some unique positive trace operator $T$ on $\overline{S}$ with unit trace. Observe that, in contrast to $L(H)$, regular states on $E(S)$ are not $\sigma$-additive. In fact, for a separable inner product space $S$, $E(S)$ admits a $\sigma$-additive state only if $S$ is complete. In Theorems 2.1 and 2.8, it was seen that the range of states on $E(S)$ can be finite. We shall show that this cannot be when our state $s$ is regular. Before showing this, we prove the following lemmas.

**Lemma 3.4.** *Let* $\{f_1, f_2, \ldots, f_n\}$ *be a finite ONS in the completion* $\overline{S}$ *of an inner product space S. For every* $\delta > 0$ *there exists an ONS* $\{h_1, h_2, \ldots, h_n\} \subset S$ *such that* $\|f_i - h_i\| < \delta$ *for all* $i \leq n$.

**Proof:** Let $M_1 = \text{span}\{f_1, f_2, \ldots, f_{n-1}\}^{\perp_{\bar{S}}}$. Since $f_n \in M_1$ and because $M_1 \cap S$ is dense in $M_1$, there exists $h_n \in M_1 \cap S$ such that $\|f_n - h_n\| < \delta$. Put $M_2 = \text{span}\{f_1, f_2, \ldots, f_{n-2}, h_n, f_n\}^{\perp_{\bar{S}}}$. Repeating the same argument, we can find $h_{n-1} \in M_2 \cap S$ such that $\|f_{n-1} - h_{n-1}\| < \delta$. Continuing like this, we construct the ONS $\{h_1, h_2, \ldots, h_n\} \subset S$ satisfying the required condition. □

**Lemma 3.5.** *Let s be a state on $E(S)$, $dim S = \infty$. There exists a unique Hermitian operator $T_0$ on $\overline{S}$ such that $s([x]) = \langle T_0 x, x\rangle$ for all $x \in \mathcal{S}(S)$. Moreover, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in \mathcal{S}(S)$ satisfying $||x - y|| < \delta$, we have $|s([x])| - s([y])| < \epsilon$.*

**Proof:** By restricting $s$ on $L(N)$, where $N$ is a three-dimensional subspace of $S$, we get a finitely additive (positive) measure $s|_{L(N)}$ on $L(N)$. By Gleason's theorem, there exists a bounded symmetric bilinear form $t_N$ on $N \times N$ such that $s([x]) = t_N(x, x)$ holds for all $x \in \mathcal{S}(N)$. Since every symmetric bilinear form is uniquely determined by its quadratic form, we can define a bilinear form $t$ on $S \times S$ as follows: for any $x, y \in S$, let $N$ be any three-dimensional subspace containing $x$ and $y$, then put $t(x, y) = t_N(x, y)$. It is clear that $t(x, x) = s([x])$ for all $x \in \mathcal{S}(S)$. Since $s$ is bounded, $t$ is also bounded and therefore $t$ can be uniquely extended to a bounded symmetric bilinear form $\overline{t}$ on $\overline{S} \times \overline{S}$. Consequently, there is a unique Hermitian operator $T_0$ on $\overline{S}$ such that $s([x]) = \langle T_0 x, x\rangle$ for all $x \in \mathcal{S}(S)$.

Now let $x, y \in S(S)$. Then we have

$$\begin{aligned}|s([x]) - s([y])| &= |\langle T_0 x, x\rangle - \langle T_0 y, y\rangle| \\ &= |\langle T_0 x, x\rangle - \langle T_0 x, y\rangle + \langle T_0 x, y\rangle - \langle T_0 y, y\rangle| \\ &< |\langle T_0 x, x - y\rangle| + |\langle T_0 x - T_0 y, y\rangle| \\ &\leq 2||T_0|| \cdot ||x - y||,\end{aligned}$$

which implies that $s$ is "continuous on $\mathcal{S}(S)$." □

We remark that it can be also shown that the Hermitian operator $T_0$ obtained in Lemma 3.5 is of trace class.

**Theorem 3.6.** *Let S be an incomplete inner product space and let T be a positive trace operator on $\overline{S}$ with unit trace. The state $s_T$ on E(S) defined as in Eq. (3.3), satisfies* $[0, 1) \subset \mathrm{Range}_f(s_T)$. *Moreover,* $1 \in \mathrm{Range}_f(s_T)$ *if, and only if, T has only a finite system of proper vectors, and these are all in S.*

**Proof:** The Hermitian trace operator $T$ can be expressed as

$$T = \sum_{i \in I} \lambda_i x_i \otimes \overline{x_i},$$

where $\{\lambda_i : i \in I\}$ are the eigenvalues (possibly repeated) of $T$ corresponding to the proper vectors $\{x_i : i \in I\}$. Moreover, since $T$ is of unit trace, we have $\sum_{i \in I} \lambda_i = 1$. This implies that for every $\epsilon > 0$, there exists a finite subset $I_0 \subset I$ such that $\sum_{i \in I_0} \lambda_i > 1 - \epsilon$. Take an ONS $\{w_i : i \in I_0\} \subset \mathrm{span}\{x_i : i \in I_0\}^{\perp_{\overline{S}}}$. Then $\sum_{i \in I_0} \langle \mathrm{T}_{\mathrm{w}_i}, w_i\rangle < \epsilon$. By Lemma 3.5 there exists $\delta > 0$ such that for each

$x_i, w_i, i \in I_0$ and $u, v \in \mathcal{S}(S)$, we have

$$|\mathcal{S}_T([u]) - \lambda_i| < \frac{\varepsilon}{I_0} \quad \text{whenever } |u - x_i| < \delta,$$

$$|\mathcal{S}_T([v]) - \langle Tw_i, w_i\rangle| < \frac{\varepsilon}{I_0} \quad \text{whenever } |v - w_i| < \delta.$$

By Lemma 3.4, we can find an ONS $\{u_i : i \in I_0\} \cup \{v_i : i \in I_0\} \subset S$ such that $||u_i - x_i|| < \delta$ and $||v_i - w_i|| < \delta$, for each $i \subset I_0$. This implies that

$$a = s_T\big(\oplus_{i\in I_0} [ui]\big) > 1 - 2\epsilon, \quad \text{and} \quad b = s_T\big(\oplus_{i\in I_0} [v_i]\big) < 2\epsilon.$$

For each $i \in I_0$, let $y_i = \cos\phi\, u_i + \sin\phi\, v_i$, where $\phi \in [0, \frac{\pi}{2}]$. Set $Y_{I_0} = \oplus_{i\in I_0}[y_i]$. Then

$$\begin{aligned}
s_T(Y_{I_0}) &= \sum_{i\in I_0} s_T([y_i]) \\
&= \sum_{i\in I_0}\sum_{j\in I} \lambda_i |\langle y_j, x_j\rangle|^2 \\
&= \sum_{i\in I_0}\sum_{j\in I} \lambda_j |\langle \cos\phi u_i + \sin\phi v_i, x_j\rangle|^2 \\
&= \sum_{i\in I_0}\sum_{j\in I} \lambda_j \{\cos^2\phi|\langle u_i, x_j\rangle|^2 + \sin^2\phi|\langle v_i, x_j\rangle|^2 \\
&\quad + \sin\phi\cos\phi\{\langle v_i, x_j\rangle\langle x_j, u_i\rangle + \langle u_i, x_j\rangle\langle x_j, v_i\rangle\}\} \\
&= a\ \cos^2\phi + b\ \sin^2\phi + \gamma\ \sin\phi\cos\phi,
\end{aligned}$$

where $\gamma \in \mathbb{R}$. By elementary real analysis theory, it follows that

$$[a, b] \subset \text{Range}_f(s).$$

This implies that $[2\epsilon, 1 - 2\epsilon] \subset \text{Range}_f(s)$. Since $\epsilon$ was arbitrary, we have $[0, 1) \subseteq \text{Range}_f(s)$. We conclude by noting that $s_T(M) = 1$ if, and only if, $\{x_i : i \in I\} \subset \overline{M}$. Thus, $1 \in \text{Range}_f(s_T)$ only when $\{x_i : i \in I\}$ is finite and is contained in $S$. □

**Corollary 3.7.** *Let S be an incomplete inner product space and let s be a regular state on E(S). For any* $A \in E(S)$, $\dim A = \infty$, *we have*

$$[0, s(A)) \subset \{s(M) : M \subset A, \dim A < \infty\} \subset [0, s(A)].$$

By considering the Aarnes decomposition of any state $s$ on $E(S)$, $s = \alpha s_R + \beta s_V$, it immediately follows (by Corollary 3.7), that $[0, \alpha) \subset \text{Range}_f(s)$. However, observe that the range of $s$ need not be convex as it is in the case when $S$ is complete.

Only if the regular component of $s$ is "sufficiently large," we can guarantee that Range$(s) = [0, 1]$. This happens when $\alpha > \frac{1}{2}$.

We have also examples of states on $E(S)$ such that the regular component of their decomposition is not zero (i.e. $\alpha \neq 0$), and yet their range is not convex. For example, let $S$ be an inner product space such that $E(S)$ consists of the finite/cofinite subspaces of $S$. (Refer to Remark 2.9.) Let $s_V$ be a state on $E(S)$ vanishing on all the finite-dimensional subspaces of $S$ (observe that in this case this state is necessarily two-valued), and let $s_R$ be any regular state on $E(S)$. Let $0 < \alpha < \frac{1}{2}$, and consider the state $s = \alpha s_R + (1-\alpha)s_V$. It is not difficult to verify that $(\alpha, 1-\alpha) \cap \text{Range}(s) = \phi$.

## 4. REGULAR CHARGES ON $L(H)$

We recall that a cardinal number $\boldsymbol{a}$ is said to be *nonmeasurable* if for every set $\mathcal{A}$ having cardinality $\boldsymbol{a}$, the power set of $\mathcal{A}$ admits no $\sigma$-additive probability measure $\mu$ satisfying $\mu(\{x\}) = 0$ for all $x \in \mathcal{A}$. (Refer to Ulam, 1930.) For each $n = 0, 1, 2, \dots,$ the cardinal $\aleph_n$ is nonmeasurable. Moreover, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are two cardianals such that $a \leq \boldsymbol{b}$ and $\boldsymbol{b}$ is nonmeasurable, then $\boldsymbol{a}$ is also nonmeasurable. In addition, if we adopt the continuum hypothesis, then the cardinality of $\mathbb{R}$, $2^{\aleph_0}$, is equal to $\aleph_1$, and therefore is also nonmeasurable. It should be noted that most practical applications involve nonmeasurable cardinals; very often countable cardinals. Moreover, there are set-theoretical models, relative to which, each cardinal is nonmeasurable.

**Theorem 4.1.** *Let H be a Hilbert space whose dimension is a nonmeasurable cardinal. Every $\sigma$-additive charge on L(H) is completely additive and regular.*

**Proof:** Let $m$ be a $\sigma$-additive charge on $L(H)$ and let $\{M_i : i \in I\}$ be any collection of a mutually orthogonal subspaces in $L(H)$. Define the mapping $\mu : 2^I \to \mathbb{R}$, $\mu(J) = m(\vee_{i \in J} M_i)$. The set function $\mu$ defines a finite signed-measure on $2^I$, and therefore it admits a Jordan decomposition, i.e. it can be expressed as the difference of two positive finite signed measures $\mu^+$ and $\mu^-$. By the Theorem of Ulam (1930), there exist two (at most) countable subsets $J_+$ and $J_-$ of $I$ such that $\mu_+(I \setminus J_+) = \mu_-(I \setminus J_-) = 0$. Put $J_0 = J_+ \cup J_-$. Then $\mu(I \setminus J_0) = \mu_+(I \setminus J_0) - \mu_-(I \setminus J_0) = 0$. This implies that

$$
\begin{aligned}
m\left(\bigvee_{i \in I} M_i\right) &= \mu(I) = \mu(J_0) + \mu(I \setminus J_0) \\
&= \sum_{i \in J_0} \mu(\{i\}) + \sum_{i \in I \setminus J_0} \mu(\{i\}) \\
&= \sum_{i \in I} m(M_i).
\end{aligned}
$$

In particular, let $M = \vee_{i\in I}[x_i]$, where $\{x_i : i \in I\}$ if an ONB of $M$. Then $m(M) = \sum_{i\in I} m([x_i])$, and therefore $m$ is regular. □

If $H$ is a Hilbert space whose dimension is a nonmeasurable cardinal, then the set of regular states on $L(H)$ coincides with the set of $\sigma$-additive states. The same cannot be said for charges. Dorofeev and Sherstnev (1990) proved that every completely additive charge on $L(H)$, dim $H = \infty$, is bounded. If we restrict ourselves to spaces with nonmeasurable dimension, we have: every $\sigma$-additive charge on $L(H)$ is bounded. Our aim is to show that for an infinite-dimensional Hilbert space $H$, there always exist a regular charge on $L(H)$ which is unbounded.

First we define a Hamel discontinuous function on $\mathbb{R}$ as follows. (See also Hamel, 1905.) Let $\mathfrak{B} = \{x_s : s \in \Sigma\}$ be a Hamel basis in $\mathbb{R}$ over the field of rational numbers. It is harmless to assume that $x_s > 0$ for each $s \in \Sigma$. Fix an element $x_{s_0} \in \mathfrak{B}$. Then every real number $x \in \mathbb{R}$ can be uniquely expressed in the form

$$x = \beta_{s_0} x_{s_0} + \sum_{s\in\sigma} \beta_s x_s, \tag{4.1}$$

where $\sigma$ is a finite subset of $\Sigma\backslash\{s_0\}$ and $\beta$'s are rational numbers. We define a Hamel discontinuous function $\phi : \mathbb{R} \to \mathbb{Q}$ by $\phi(x) = \beta_{s_0}$ whenever $x \in \mathbb{R}$ is of the form (4.1).

Let $s$ be any regular state on $L(H)$. We claim to show that $\phi$ o s is a regular charge. Let $\epsilon > 0$ and $A \in L(H)$ be given. If $\phi(s(A)) = 0$, we take $M = \{0\}$, which yields $|\phi(s(A)) - \phi(s(M))| < \epsilon$. So let $0 \neq s(A) = \beta_{s_0} x_{s_0} + \sum_{s\in\sigma} \beta_s x_s$ where $\beta s_0 \neq 0$. There is an integer $n \geq 1$ such that $1/n < \epsilon$ and $x_{s_0}/n < s(A)$. Then $0 < (\beta_{s_0} - 1/n)x_{s_0} + \sum_{s\in\sigma} \beta_s x_s < s(A)$. By Corollary 3.2, there is a finite-dimensional subspace $M$ of $A$ such that $s(M) = (\beta_{s_0} - 1/n)x_{s_0} + \sum_{s\in\sigma} \beta_s x_s$. Hence, $|\phi(s(A)) - \phi(s(M))| = 1/n < \epsilon$ which proves that $\phi \circ s$ is a regular charge on $L(H)$.

In Chetcuti and Dvurečenskji (2003), the authors have proved that the range of a bounded charge on $L(H)$ is always convex in $\mathbb{R}$. This implies that the charge $\phi \circ s$ is unbounded. In view of the Dorofeev–Sherstnev result, it follows that $\phi \circ s$ is not $\sigma$-additive.

**Theorem 4.2.** *Let H be a Hilbert space whose dimension is an infinite nonmeasurable cardinal. The set of $\sigma$-additive charges on L(H) is a proper subset of the set of regular charges on L(H).*

Observe also that $\text{Range}_f(\phi \circ s)$ is rationally convex.[5] Indeed, let $\beta_1 = \phi(s(A)) < \phi(s(B)) = \beta_2$, where $A, B \in L(H)$ and let $\beta \in \mathbb{Q}$ such that[5] $\beta_1 < \beta <$

[5] A subset $A$ of $\mathbb{R}$ is *rationally convex* if for any $x_1, x_2 \in A$, $\lambda \in \mathbb{Q} \cap (0, 1)$, we have $\lambda x_1 + (1 - \lambda)x_2 \in A$.

$\beta_2$ be given. There is a rational number $\lambda$, $0 < \lambda < 1$, such that $\beta = \lambda\beta_1 + (1 - \lambda)\beta_2$. Therefore $x = \lambda s(A) + (1-\lambda)s(B) \in [0, 1)$. By Theorem 3.2, there is a finite-dimensional subspace $M \in L(H)$ such that $s(M) = x$. Consequently, $\beta = \phi(s(M)) \in \text{Range}_f(\phi \circ s)$. Since $\phi \circ s$ is unbounded, we have $\text{Range}_f(\phi \circ s) = \text{Range}(\phi \circ s) = \mathbb{Q}$.

We recall that a charge $m$ on $E(S)$ (or on $F(S)$) is $P(S)$-*bounded* (resp. $P_1(S)$-bounded) if $\text{Range}_f(m)$ is bounded (resp. $\text{Range}_1(m)$ is bounded). If in the previous construction we choose our regular state to be a vector state $s_u$, for some $u \in S(H)$, one can easily verify that $\text{Range}_1(\phi \circ s_u)$ is unbounded. Thus, not every regular charge on $L(H)$ is $P_1(H)$-bounded. This answers to the negative a question asked in Chetcuti and Dvurečenskij (2004), whether every regular charge on $F(S)$ is $P_1(S)$-bounded.

In Chetcuti and Dvurečenskij (2003) and Chetcuti and Dvurečenskij (2004), the notion of sign-preserving charges was introduced. In Chetcuti and Dvurečenskij (2004), it was proved that every regular sign-preserving charge on $L(H)$ is $\sigma$-additive. From this, and from the above discussion, we see that the regularity of a charge is not sufficient for it to satisfy the sign-preserving property.

We conclude by giving a sufficient condition for a regular charge on *L(H)* to be $\sigma$-additive.

**Theorem 4.3.** *Let H be an infinite-dimensional Hilbert space and let m be a $P_1(H)$-bounded, regular charge on L(H). Then m is $\sigma$-additive. Moreover,* $\text{Range}_f(m)$ *contains $(\alpha, \beta)$, where* $\alpha = \inf\{m(A) : A \in L(H)\}$ *and* $\beta = \sup\{m(A) : A \in L(H)\}$.

**Proof:** We can repeat the steps of the proof of Lemma 3.5 to obtain a Hermitian operator $T$ on $H$ satisfying that $m([x]) = \langle Tx, x\rangle$ for all $x \in \mathcal{S}(H)$.

We show that $T$ is a trace class operator. First we recall that $T$ can be expressed as the difference of two positive operators $T_1$ and $T_2$, and $H$ can be split into two orthogonal subspaces $H_1$ and $H_2$ such that $T_1 H_2 = T_2 H_1 = 0$. Since $T_1$ is positive, to show that it is a trace operator, it is sufficient to verify that $\sum_{i\in I}\langle T_1 x_i, x_i\rangle$ is summable for one ONB $\{x_i : i \in I\}$ in $H$. Let $\{x_i : i \in I_0\}$ and $\{y_j : j \in J_0\}$ be orthonormal bases of $H_1$ and $H_2$ respectively. Then

$$\sum_{i\in I_0}\langle T_1 x_i, x_i\rangle + \sum_{j\in J_0}\langle T_1 y_j, y_j\rangle = \sum_{i\in I_0}\langle T_1 x_i, x_i\rangle = \sum_{i\in I_0} m([x_i]). \tag{4.2}$$

Since $m$ is regular, and because $m$ is positive on all the finite-dimensional subspaces of $H_1$, it follows that $m$ is positive (and therefore monotone) on $L(H_1)$. Hence, for any finite subset $I_0'$ of $I_0$, we have $0 \le \sum_{i\in I_0'} m([x_i]) \le m(H_1)$. This implies that $0 \le \sum_{i\in I_0} m([x_i]) \le \infty$, and therefore $T_1$ is a trace operator. The same can be shown for $T_2$, and therefore it follows that $T$ is a trace operator. So we can define the completely additive charge $m_T$ on $L(H)$ by setting $m_T(M) = \mathrm{tr}(TP_M)$, $M \in L(H)$. Our goal is to show that $m_\top = m$. Since $m$ and $m_T$ are regular and monotone on $L(H_1)$ and $L(H_2)$, it follows that $m(H_i) = m_T(H_i)$, $i = 1, 2$. Hence,

$$m(H) = m(H_1) + m(H_2) = m_T(H_1) + m_T(H_2) = tr(TP_H).$$

Now let $M \in L(H)$. If we restrict $m$ to $L(M)$, we can derive a Hermitian trace operator $T_M$ on $M$ such that

$$\langle T_M x, x\rangle = m([x]) = \langle Tx, x\rangle$$

for all $x \in \mathcal{S}(M)$. Since bilinear forms are uniquely determined by their corresponding quadratic forms, it follows that $\langle T_M x, y\rangle = \langle Tx, y\rangle$ for all $x$,$y \in M$, and therefore, $\mathrm{tr}(T_M) = \mathrm{tr}(T P_M)$. (Observe that $T_M$ is equal to the restriction of $P_M TP_M$ on $M$.) The subspace $M$ can be split into two orthogonal subspces $M_1$ and $M_2$ such that $T_M$ is positive on $M_1$ and negative on $M_2$. Thus we can repeat the same arguments as in the preceding paragraph and we simply interchange $H$ with $M$ and $T$ with $T_M$, and we get $m(M) = \mathrm{tr}(T_M)$, which implies that $m(M) = \mathrm{tr}(TP_M)$.

Now we show that $(\alpha, \beta) \subset \mathrm{Range}_f(m)$. For every $\epsilon > 0$, there exists a finite-dimensional subspace $A \subset H$ such that $m(A) > \beta - \epsilon$. In addition, we can find a finite-dimensional subspace $A' \subset A^\perp$ such that $\dim A' = \dim A$ and $-\epsilon < m(A') < \epsilon$. If we let $\{a_i : i \leq n\}$ and $\{a_i' : i \leq n\}$ be orthonormal bases in $A$ and $A'$ respectively, we can define $y_i = \cos\phi a_i + \sin\phi a_i'$, $\phi \in [0, \frac{\pi}{2}]$, and $Y = \oplus_{i\leq n}[y_i]$. By a similar argument to that used in the proof of Theorem 3.6, one can show that $[m(A'), m(A)] \subset \mathrm{Range}_f(m)$. This implies that $[0, \beta] \subset \mathrm{Range}_f(m)$. But we can also find a finite-dimensional subspace $B \subset H$ such that $m(B) < \alpha + \epsilon$. By repeating the same arguments of above, we deduce that $(\alpha, 0] \subset \mathrm{Range}_f(m)$. Thus $(\alpha, \epsilon) \subset \mathrm{Range}_f(m)$. □

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